A treatment on Larmor Precession and Rabi resonance

Contents

Larmor Precession

The Larmor precession is caused by an external magnetic field (B-field) on the magnetic moment (μ) original from the total angular momentum (J). The angular momentum J is from two sources, orbital angular momentum and spin.

Angular Momentum

$$J = L + S = (J_x, J_y, J_z)$$

Magnetic moment

$$\mu = g\frac{\mu_B}{\hbar} J = \gamma J$$

The Hamiltonian for the external B field is point alone positive z-axis.

$$B = (0, 0, B_0)$$

$$H = -\mu \cdot B = -\gamma J \cdot B = -\gamma B_0 J_z$$

The Schrodinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle = -\gamma B_0 J_z|\psi(t)\rangle$$

$$|\psi(t)\rangle = Exp(\frac{i}{\hbar} \gamma B_0 t J_z)|\psi(0)\rangle$$

Recall the right-hand rotational operator on z-axis

$$R_z(\theta) = Exp(-\frac{i}{\hbar}\theta J_z)$$

Thus, we can re write the solution as

$$|\psi(t)\rangle = R_z(-\gamma B_0 t)|\psi(0)\rangle = R_z(\omega_0 t)|\psi(0)\rangle$$

Which mean, the state $|\psi(0)\rangle$ is rotating on right –hand along the z-axis with angular velocity $(-\gamma B_0)$. We define

$$\omega_0 = -\gamma B_0 = Larmor\ frequency$$

Such that

$$\omega_0 > 0 \Leftrightarrow rotating\ right\ hand\ \Leftrightarrow \gamma < 0$$

$$\omega_0 < 0 \Leftrightarrow rotating\ left\ hand\ \Leftrightarrow \gamma > 0$$

Thus, we also use another terminology on rotation direction

$$right\ hand\ \Leftrightarrow positive\ direction$$

$$left\ hand\ \Leftrightarrow negative\ direction$$

And in the following, we will use $\omega_0 > 0$.

In order to see the rotation, we have to find the expectation of J, or the magnetization or the spin polarization. In classical analogy, it is equivalent to the Bloch vector.

$$M = \langle J\rangle = \left(\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle\right) = (M_x, M_y, M_z)$$

$$\langle J_x\rangle_t = \langle R_z(-\omega_0 t)\, J_x\, R_z(\omega_0 t)\rangle_{t=0} = \langle J_x\rangle_{t=0} \cos(\omega_0 t) - \langle J_y\rangle_{t=0} \sin(\omega_0 t)$$

$$\langle J_y\rangle_t = \langle R_z(-\omega_0 t)\, J_y\, R_z(\omega_0 t)\rangle_{t=0} = \langle J_x\rangle_{t=0} \sin(\omega_0 t) + \langle J_y\rangle_{t=0} \cos(\omega_0 t)$$

$$\langle J_z \rangle_t = \langle R_z(-\omega_0 t)\, J_z\, R_z(\omega_0 t) \rangle_{t=0} = \langle J_z \rangle_{t=0}$$

We can see the magnetization rotate around the z-axis by $\omega_0 t$

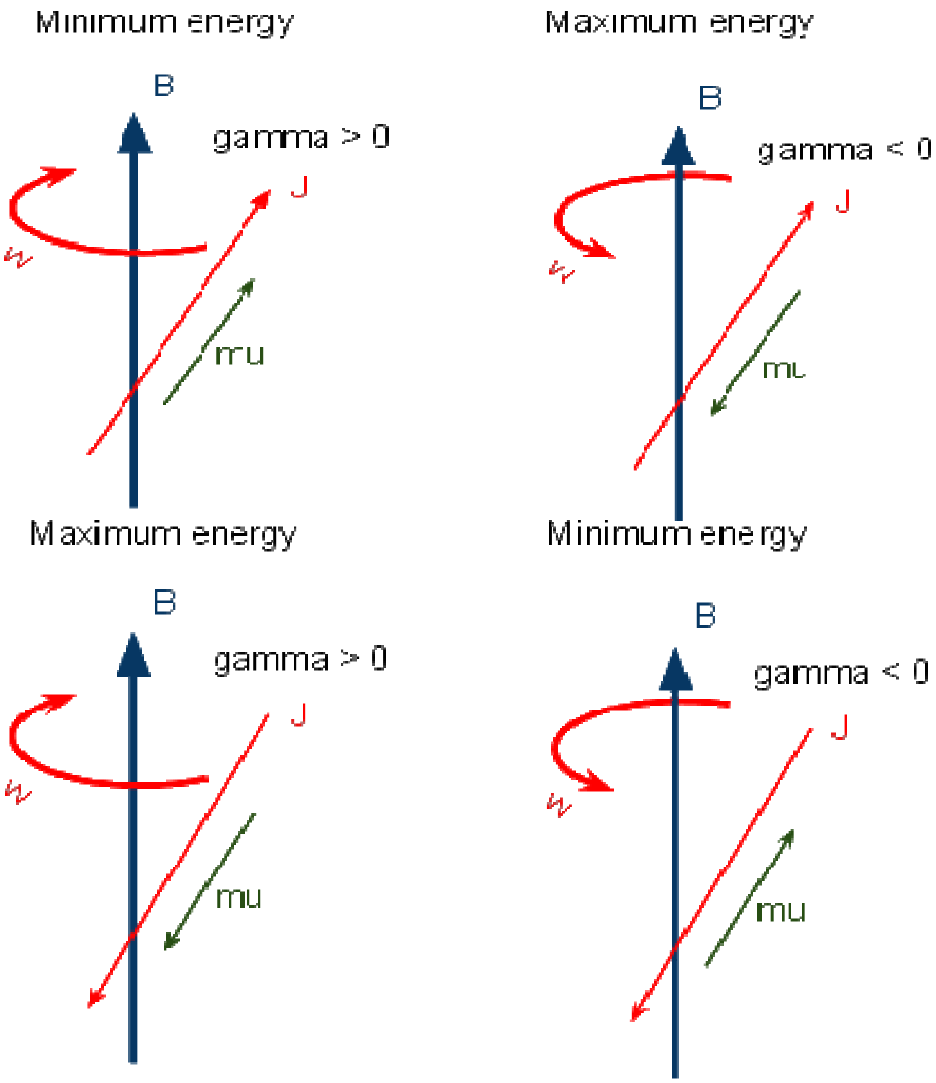
$$M(t) = \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} M(0)$$

this is just an ordinary rotation in real space.

Here we listed the relation on μ, γ, J, and rotating direction:

$\mu \rightrightarrows B$	$\mu = \gamma J$	$H = -\mu \cdot B$	$R_z(-\gamma B_0 t)$
>0	parallel	Min	negative
<0	anti-parallel	Min	positive

$\mu \rightleftarrows B$	$\mu = \gamma J$	$H = -\mu \cdot B$	$R_z(-\gamma B_0 t)$
>0	parallel	Max	negative
<0	anti-parallel	Max	positive



Notice: some reader may confuse on the rotation direction. How come the J operator rotates backward but the expectation value moving forward? We can think that, the operator are stick with the coordinate, in fact, the operator define the coordinate axis. Thus, the coordinate axis moving backward, the coordinate are moving forward.

Spin ½

For 2 states system, the spin function is

$$\chi(0) = \langle x|\psi(0)\rangle = (c_1\, , Exp(i\theta)c_2)$$

$$\chi(t) = \langle x|\psi(t)\rangle = (\, Exp\left(-i\frac{\omega_0}{2}t\right) c_1\, , Exp\left(i\frac{\omega_0}{2}t + i\theta\right) c_2)$$

Thus the population never changes.

The magnetization is:

$$M(0) = \frac{\hbar}{2}(2c_1c_2\cos(\theta), 2c_1c_2\sin(\theta), c_1^2 - c_2^2)$$

$$M(t) = \frac{\hbar}{2}(2c_1c_2\cos(\omega_0 t + \theta), 2c_1c_2\sin(\omega_0 t + \theta), c_1^2 - c_2^2)$$

For a mixed state, it will rotate on positive direction.

For a pure state, the transverse components are all zero.

Spin 1

For 3 states system, the spin function is

$$\chi(0) = \langle x|\psi(0)\rangle = (c_1, c_2Exp(i\alpha), c_3Exp(i\beta))$$

$$M(0) = \hbar\begin{pmatrix} \sqrt{2}c_2(c_1\cos(\alpha) + c_3\cos(\alpha - \beta)) \\ \sqrt{2}c_2(c_1\sin(\alpha) - c_3\sin(\alpha - \beta)) \\ c_1^2 - c_3^2 \end{pmatrix}$$

$$\chi(t) = \langle x|\psi(t)\rangle = (c_1Exp(-i\omega_0 t), c_2Exp(i\alpha), c_3Exp(i\omega_0 t + i\beta))$$

$$M(t) = \hbar\begin{pmatrix} \sqrt{2}c_2(c_1\cos(\omega t + \alpha) + c_3\cos(\omega t - \alpha + \beta)) \\ \sqrt{2}c_2(c_1\sin(\omega t + \alpha) + c_3\sin(\omega t - \alpha + \beta)) \\ c_1^2 - c_3^2 \end{pmatrix}$$

Density Matrix

There is other way to visualize the effect of the spin polarization beside of the magnetization, by the density matrix.

The density matrix for is defined as

$$\rho = [\rho_{ij}], \quad \rho_{ij} = \chi_i^*\chi_j$$

The size of the density matrix is determined by the size of the spinor.

Notice: the density matrix is taken not on single spin but on ensemble, thus, the average is automatically applied.

$$\rho_{ij} = \frac{1}{N} \sum_{all\ spin} \chi_i^* \chi_j$$

Where N is total number of spin.

The diagonal elements are called population

$$\rho_{ii} = |\chi_i|^2$$

Which is the population of state i.

The other elements are called coherent. When there is a transverse component of the magnetization, it will be reflected at those elements. In order to have a transverse magnetization, the spins have to be point in same direction. I guess the name is come from this. If the spins are not coherent, they are pointing in different direction, the coherency lost.

The density matrix is Hermitinian, which mean, it can be measure.

Spin ½

For single 2 state systems, the possibility of it in up-state or down-sate is

$$\chi = \left(c_1, c_2 Exp(i\,\theta) \right), for\ c_1, c_2, \theta \in Reals$$

$$c_1^2 + c_2^2 = 1$$

The density matrix is

$$\rho = \begin{pmatrix} \rho_+ & \rho_c \\ \rho_c^* & \rho_- \end{pmatrix} = \begin{pmatrix} c_1^2 & c_1 c_2 Exp(i\,\theta) \\ c_1 c_2 Exp(-i\,\theta) & c_2^2 \end{pmatrix}$$

The component of the density matrix can be interpolated in this way:

$$\rho_+ = |+\rangle\ population$$

$$\rho_- = |-\rangle\ population$$

$$\rho_c = coherent$$

Notice that the populations are along the z-axis, thus the total magnetization density on z-axis is:

$$M_z = \rho_+ - \rho_-$$

The coherent is a historical name, in fact, it reflect the degree of transverse magnetization. To see what exactly the coherent is, let's check the consistency with $\langle J \rangle$, which is the magnetization for single spin.

$$M = \frac{\hbar}{2}(2c_1c_2\cos(\theta), 2c_1c_2\sin(\theta), c_1^2 - c_2^2)$$

Or

$$M = \frac{\hbar}{2}(2Re(\rho_c), 2Im(\rho_c), \rho_+ - \rho_-)$$

Or

$$M = \frac{\hbar}{2}(\rho_c + \rho_c^*, \rho_c - \rho_c^*, \rho_+ - \rho_-)$$

Notice that

$$\rho_+ + \rho_- = 1$$

Thus all combination of the density matrix has a real meaning.

$\rho_+ + \rho_- = 1$	Total population
$\rho_+ - \rho_- = M_z$	Longitudinal Magnetization
$\rho_c + \rho_c^* = M_x$	Cosine of Transverse Magnetization
$i(\rho_c - \rho_c^*) = M_y$	Sine of Transverse Magnetization

Thus, the direction of the magnetization is given by the phase different of up state and down state. And we can see the density matrix is equivalent representation of the spin polarization. For a state with

$$\chi = (c_1, c_2) = (1, 0)$$

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M = \frac{\hbar}{2}(0, 0, 1)$$

In the case of Larmor precession, the state is:

$$\chi(0) = \langle x|\psi(0)\rangle = (c_1 \, , Exp(i\theta)c_2)$$

$$\chi(t) = \langle x|\psi(t)\rangle = (\, Exp\left(-i\frac{\omega_0}{2}t\right)c_1 \, , Exp\left(i\frac{\omega_0}{2}t + i\theta\right)c_2)$$

$$\rho(0) = \begin{pmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{pmatrix}$$

$$\rho(t) = \begin{pmatrix} c_1^2 & c_1c_2Exp(i\,\omega\,t + i\,\theta) \\ c_1c_2Exp(-i\,\omega\,t - i\,\theta) & c_2^2 \end{pmatrix}$$

$$M(t) = \frac{\hbar}{2}(2c_1c_2\cos(\omega_0 t + \theta)\, , 2c_1c_2\sin(\omega_0 t + \theta)\, , c_1^2 - c_2^2)$$

Thus, for a mixed state, it rotate right hand for $\omega_0 > 0 \;\Leftrightarrow \gamma < 0$

If there is not magnetization,

$$M = 0 \; or \; \rho = 0 \;\Rightarrow c_1 = c_2 = 0$$

Suppose there are N spins,

$$\chi_+ = \frac{1}{N}\sum_{all} c_1 \, , \chi_- = \frac{1}{N}\sum_{all} c_2 \, , \qquad (c_1, c_2) \in Complex$$

It is possible to have the average spinor, which norm is smaller than 1.

Spin 1

For a single 3 state system, the spinor has 3 components

$$\chi(0) = \langle x|\psi(0)\rangle = (c_1 \, , c_2Exp(i\alpha), c_3Exp(i\,\beta))$$

$$\rho(0) = \begin{pmatrix} c_1^2 & c_1c_2Exp(i\,\alpha) & c_1c_3Exp(i\,\beta) \\ c_1c_2Exp(-i\,\alpha) & c_2^2 & c_2c_3Exp(-i\,\alpha + i\beta) \\ c_1c_3Exp(-i\,\beta) & c_2c_3Exp(i\,\alpha - i\beta) & c_3^2 \end{pmatrix}$$

$$M(0) = \hbar\begin{pmatrix} \sqrt{2}c_2(c_1\cos(\alpha) + c_3\cos(\alpha - \beta)) \\ \sqrt{2}c_2(c_1\sin(\alpha) - c_3\sin(\alpha - \beta)) \\ c_1^2 - c_3^2 \end{pmatrix}$$

$$\chi(t) = \langle x|\psi(t)\rangle = (\, c_1Exp(-i\omega\,t)\, , c_2Exp(i\alpha), c_3Exp(\,i\,\omega\,t + i\beta))$$

$$\rho(t) = \begin{pmatrix} c_1^2 & c_1 c_2 Exp(-i\omega t + i\,\alpha) & c_1 c_3 Exp(i\,\beta) \\ c_1 c_2 Exp(i\omega t - i\,\alpha) & c_2^2 & c_2 c_3 Exp(i\omega t - i\,\alpha + i\beta) \\ c_1 c_3 Exp(-i\,\beta) & c_2 c_3 Exp(-i\omega t + i\,\alpha - i\beta) & c_3^2 \end{pmatrix}$$

$$M(t) = \hbar \begin{pmatrix} \sqrt{2} c_2 (c_1 \cos(\omega t + \alpha) + c_3 \cos(\omega t - \alpha + \beta)) \\ \sqrt{2} c_2 (c_1 \sin(\omega t + \alpha) + c_3 \sin(\omega t - \alpha + \beta)) \\ c_1^2 - c_3^2 \end{pmatrix}$$

Lets investigate what the components of the density matrix means.

For a spin-1 system, the diagonal elements are still showing the population of each state. But the coherent are little bit complicated. For simplicity, we set the phase different on each state become zero.

$$\rho(t) = \begin{pmatrix} c_1^2 & c_1 c_2 Exp(-i\omega t) & c_1 c_3 \\ c_1 c_2 Exp(i\omega t) & c_2^2 & c_2 c_3 Exp(i\omega t) \\ c_1 c_3 & c_2 c_3 Exp(-i\omega t) & c_3^2 \end{pmatrix}$$

And denote it

$$\rho(t) = \begin{pmatrix} \rho_+ & \rho_{+0} & \rho_{+-} \\ \rho_{+0}^* & \rho_0 & \rho_{0-} \\ \rho_{+-}^* & \rho_{0-}^* & \rho_- \end{pmatrix}$$

And compare it to the magnetization

$$M(t) = \hbar \begin{pmatrix} \sqrt{2} c_2 \cos(\omega t)\,(c_1 + c_3) \\ \sqrt{2} c_2 \sin(\omega t)\,(c_1 + c_3) \\ c_1^2 - c_3^2 \end{pmatrix}$$

Since the $|1,0\rangle$ state does not have any projection on the z-axis, the longitudinal magnetization is just

$$M_z = c_1^2 - c_3^2 = \rho_+ - \rho_-$$

The transverse magnetization must be given by the coherent. The transverse magnetization is:

$$M_T = \left(M_x, M_y\right) = \hbar\sqrt{2} c_2 (c_1 + c_3)(\cos(\omega t)\,, \sin(\omega t))$$

Or related to the density matrix elements

$$M_T = \frac{1}{2}\left(Re(\rho_{+0} + \rho_{-0}), Im(-\rho_{+0} + \rho_{-0})\right)$$

Small summery

$$|\psi(0)\rangle = \sum |m\rangle c_m \xrightarrow{R_z(\omega t)} |\psi(t)\rangle = \sum R_z(\omega t)|m\rangle c_m$$

$$\chi_i(0) = c_i \xrightarrow{R_z(\omega t)} \chi_i(t) = Exp(\omega t)c_i$$

$$\rho_{ij}(0) = \chi_i^*(0)\chi_j(0) \xrightarrow{R_z(\omega t)} \rho_{ij}(t) = \chi_i^*(t)\chi_j(t)$$

$$M_i(0) = \langle\psi(0)|J_i|\psi(0)\rangle \xrightarrow{R_z(\omega t)} M_i(0) = \langle\psi(t)|J_i|\psi(t)\rangle$$

The transformation of the density matrix can be revealed by those relationships.

$$\rho(t) = R_z(-\omega t)\rho(0)R_z(\omega t)$$

Rabi Resonance

Since the angular momentum is rotating at positive direction for $\omega_0 > 0$. Thus, our perpendicular B-field should be applied on positive direction.

$$B_R = (B_R \cos(\omega\, t)\,, B_R \sin(\omega\, t)\,, B_0)$$

The Hamiltonian

$$H_B = -\mu \cdot B_R = -\gamma B_R\left(J_x \cos(\omega\, t) + J_y \sin(\omega\, t)\right) - \gamma B_0 J_z$$

or , in rotation operator

$$H_B = \omega_R R_z(\omega\, t) J_x R_z(-\omega\, t) + \omega_0 J_z$$

and it make a very good sense, since

$$R_z(\omega\, t) J_x R_z(-\omega\, t)$$

represents Jx rotating in positive direction.

But solving this is very inconvenient, but still possible.

Rotating Frame

If we change our frame to the rotating frame that move with the rotating B-field, we can make the Hamiltonian be time-independent again.

The rotating state is related with the static state by:

$$|\psi_R\rangle = R_z(-\omega t)|\psi\rangle$$

The find out the Schrodinger equation in rotating frame, we differentiate the ket.

$$i\hbar\frac{d}{dt}|\psi_R\rangle = i\hbar\frac{d}{dt}R_z(-\omega t)|\psi\rangle + R_z(-\omega t)i\hbar\frac{d}{dt}|\psi\rangle$$

$$i\hbar\frac{d}{dt}|\psi_R\rangle = -\omega J_z R_z(-\omega t)|\psi\rangle + R_z(-\omega t)H|\psi\rangle$$

$$i\hbar\frac{d}{dt}|\psi_R\rangle = -\omega J_z|\psi_R\rangle + R_z(-\omega t)HR_z(\omega t)|\psi_R\rangle = H_R|\psi_R\rangle$$

$$H_R = R_z(-\omega t)HR_z(\omega t) - \omega J_z$$

The first term on the right hand side is just rotating the Hamiltonian on negative direction, since the rotating frame is moving in positive direction respect to the static frame. And the Hamiltonian sticks with the static frame. And the 2nd term rises by the rotating frame. The rotating frame can see an extra B-field with amplitude $\omega = -\gamma\, B_{extra}$.

Solving Larmor precession in rotating frame

The static Hamiltonian is

$$H_B = \omega_0 J_z$$

$$H_R = R_z(-\omega t)H_B R_z(\omega t) - \omega J_z$$

$$H_R = R_z(-\omega t)(\omega_0 J_z)R_z(\omega t) - \omega J_z$$

$$H_R = (\omega_0 - \omega)\, J_z$$

Thus, when $\omega_0 = \omega$, the rotating frame is moving with the spin, it see a static spin.

Solving Transverse B-field in rotating frame

The static Hamiltonian is

$$H_B = \omega_R R_z(\omega\ t) J_x R_z(-\omega\ t) + \omega_0 J_z$$

Under the rotating frame, the rotating Hamiltonian is:

$$H_R = R_z(-\omega t) H_B R_z(\omega t) - \omega J_z$$

$$H_R = R_z(-\omega t)(\omega_R R_z(\omega\ t) J_x R_z(-\omega\ t) + \omega_0 J_z) R_z(\omega t) - \omega J_z$$

$$H_R = \omega_R J_x + (\omega_0 - \omega)\ J_z$$

Thus, when $\omega_0 = \omega$, the angular momentum is rotating on x-axis with frequency

$$\omega_R = Rabi\ Frequency$$

For $\omega_0 \neq \omega$, the angular momentum is rotating on the shifted axis

$$\omega_R J_x + (\omega_0 - \omega)\ J_z$$

with effective frequency

$$\omega_{eff} = \sqrt{\omega_R^2 + (\omega_0 - \omega)^2}$$

Spin ½

We have to solve the state to find the probability of the angular momentum on the each state. (Be careful, cannot use $Exp(-\frac{i}{\hbar}(\omega_R J_x + +(\omega_0 - \omega)\ J_z)$ to this case, since Jx and Jz does not commute)

First, we find of the Hamiltonian in |+> and |−> basic.

$$\langle + | H_R | + \rangle = \omega_R \langle + | J_x | + \rangle + (\omega_0 - \omega) \langle + |\ J_z | + \rangle = \frac{\hbar}{2}(\omega_0 - \omega)$$

$$\langle + | H_R | - \rangle = \omega_R \langle + | J_x | - \rangle + (\omega_0 - \omega) \langle + |\ J_z | - \rangle = \frac{\hbar}{2}\omega_R$$

$$\langle - | H_R | + \rangle = \omega_R \langle - | J_x | + \rangle + (\omega_0 - \omega) \langle - |\ J_z | + \rangle = \frac{\hbar}{2}\omega_R$$

$$\langle - | H_R | - \rangle = \omega_R \langle - | J_x | - \rangle + (\omega_0 - \omega) \langle - |\ J_z | - \rangle = -\frac{\hbar}{2}(\omega_0 - \omega)$$

$$H_R = \frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & \omega_R \\ \omega_R & -\omega_0 + \omega \end{pmatrix}$$

The initial condition is all up-state at the beginning.

$$\chi_R(0) = \langle x|\psi_R(0)\rangle = (1\,,0)$$

to solve this, there are at least 3 methods, one is diagonalizable the matrix, solve the coupled equations directly, and using time dependent eigen state as basic and find out the coupled equations.

The solution is:

$$\chi_R(t) = \left(\cos\left(\frac{1}{2}\omega_{eff}t\right) - i\frac{\omega_0 - \omega}{\omega_{eff}}\sin\left(\frac{1}{2}\omega_{eff}t\right), -i\frac{\omega_R}{\omega_{eff}}\sin\left(\frac{1}{2}\omega_{eff}t\right)\right)$$

the probability of going to down state or spin flip is

$$|\chi_{R-}(t)|^2 = \left(\frac{\omega_R}{\omega_{eff}}\right)^2 \sin^2\left(\frac{1}{2}\omega_{eff}t\right)$$

The magnetization is:

$$M_R = \frac{\hbar}{2}\begin{pmatrix} \frac{2(\omega_0 - \omega)\omega_R}{\omega_{eff}^2}\sin^2\left(\frac{1}{2}\omega_{eff}t\right) \\ -\frac{\omega_R}{\omega_{eff}}\sin\left(\omega_{eff}t\right) \\ \frac{1}{\omega_{eff}^2}\left((\omega_0 - \omega)^2 + \omega_R^2\cos\left(\omega_{eff}t\right)\right) \end{pmatrix}$$

Which trace a circle around the shifted axis.

For $\omega_0 = \omega,\ \omega_{eff} = \omega_R$

$$M_R = \frac{\hbar}{2}(0, -\sin(\omega_R t)\,, \cos(\omega_R t))$$

Which is rotate in right hand around x-axis.

To find the state in Lab frame, using

$$|\psi\rangle = R_z(\omega t)|\psi_R\rangle$$

which is simply adding a phase.

$$\chi(t) = \begin{pmatrix} \left(\cos\left(\frac{1}{2}\omega_{eff}t\right) - i\frac{\omega_0 - \omega}{\omega_{eff}}\sin\left(\frac{1}{2}\omega_{eff}t\right)\right) Exp\left(-\frac{i}{\hbar}\omega t\right) \\ -i\frac{\omega_R}{\omega_{eff}}\sin\left(\frac{1}{2}\omega_{eff}t\right) Exp\left(\frac{i}{\hbar}\omega t\right) \end{pmatrix}$$

The magnetization in the Lab frame:

$$M = \frac{\hbar}{2}\begin{pmatrix} \frac{2\omega_R}{\omega_{eff}^2}\sin\left(\frac{1}{2}\omega_{eff}t\right)\left(\omega_{eff}\cos\left(\frac{1}{2}\omega_{eff}t\right)\sin(\omega t) + (\omega_0 - \omega)\sin\left(\frac{1}{2}\omega_{eff}t\right)\cos(\omega t)\right) \\ \frac{\omega_R}{\omega_{eff}^2}\left(2(\omega_0 - \omega)\sin^2\left(\frac{1}{2}\omega_{eff}t\right)\sin(\omega t) - \omega_{eff}\sin(\omega_{eff}t)\cos(\omega t)\right) \\ \frac{1}{\omega_{eff}^2}\left((\omega_0 - \omega)^2 + \omega_R^2\cos(\omega_{eff}t)\right) \end{pmatrix}$$

when under Rabi resonance, $\omega_0 = \omega$

$$M = \frac{\hbar}{2}\begin{pmatrix} \sin(\omega t)\sin(\omega_{eff}t) \\ -\sin(\omega_R t)\cos(\omega t) \\ \cos(\omega_R t) \end{pmatrix}$$

which is the oscillation of 2 frequencies - the Lissajous curve on a sphere.

Relaxation

If we only switch on the transverse magnetic field for some time (τ), after the field is off, the system will go back to the thermal equilibrium. it is due to the system is not completely isolated.

Instead of consider a single spin, we have to consider the ensemble. And an ensemble is described by the density matrix or the magnetization. This is visible effect by the spin.

The reason for not consider a single spin state is, we don't know what is going on for individual spin. in fact, in the previous section, the magnetization is a Marco effect. a single spin cannot have so many states, it can only have 2 states - up or down. if we insist the above calculation is on one spin, thus, it only give the chance for having that direction of polarization, which is obtained from many measurements.

So, for a single spin, the spin can only have 2 states. And if the $\omega_0 \neq \omega$, or $\omega_{eff}\,\tau \neq \pi$, the spin has chance to go to the other state, which probability is given by above. And when it goes to relax back to the minimum energy state, it will emit a photon. But when it happens, we don't know, it is a complete random process.

However, an ensemble, a collection of spins, we can have some statistic on it. for example, the relaxation time, T_1 and T_2.

Any decay is represented by

$$Exp\left(-\frac{t}{T}\right)$$

where T is relaxation time.

since T_1 is the longitudinal relaxation time and T_2 is transverse relaxation time.

thus, if we set the lowest energy state is the rest state.

$$\rho_+(t) = 1 - (1 - \rho_+)Exp\left(-\frac{t}{T_1}\right)$$

$$\rho_-(t) = \rho_- Exp\left(-\frac{t}{T_1}\right)$$

$$\rho_c(t) = \rho_c Exp\left(-\frac{t}{T_2}\right)$$

The magnetization becomes:

$$M = \frac{\hbar}{2}\left(2\,Re(\rho_c)Exp\left(-\frac{t}{T_2}\right), 2\,Im(\rho_c)Exp\left(-\frac{t}{T_2}\right), 1 - (1 - \rho_+ + \rho_-)Exp\left(-\frac{t}{T_1}\right)\right)$$