On Vector transform and Frame transform

Contents

Terminology/Notation

Few things have to say in advance.

1. A **Vector** is NOT its **coordinate**
2. A vector can only be coordinated when there is a **frame**.
3. A frame is a set of **"reference" vectors**, which span the whole space. Those reference vectors are called **basis** of a frame.
4. a **transformation** is on a vector or its coordinate. And it can be represented by a **matrix**.
5. A Matrix should act on a coordinate or basis, but not a vector.

For a basis $\hat{\alpha}$, it has a set of reference vectors and form a column vector:

$$\hat{\alpha} = \begin{pmatrix} \widehat{\alpha_1} \\ \vdots \\ \widehat{\alpha_n} \end{pmatrix}$$

We impose some properties on the basis:

1. Unitary

$$\widehat{\alpha_\iota}.\widehat{\alpha_\iota} = 1$$

2. Orthogonality

$$\widehat{\alpha_\iota}.\widehat{\alpha_\jmath} = \delta_{ij}$$

A Vector in space can be coordinated based on the basis. (the actually procedure of coordination is artificial)

$$\vec{U} = u_i^\alpha \widehat{\alpha_\iota}$$

Where the coordinate can form row a vector:

$$\overrightarrow{u_\alpha} = (u_1^\alpha, u_2^\alpha, \dots., u_n^\alpha)$$

Thus, a vector can be rewritten as:

$$\vec{U} = \overrightarrow{u_\alpha} \cdot \hat{\alpha} = (u_1^\alpha, u_2^\alpha, \dots., u_n^\alpha) \begin{pmatrix} \widehat{\alpha_1} \\ \vdots \\ \widehat{\alpha_n} \end{pmatrix}$$

Frame transform

When there are 2 frames, the same vector can be coordinated in 2 frames with different coordinate.

$$\vec{U} = \overrightarrow{u_\alpha} \cdot \hat{\alpha} = \overrightarrow{u_\beta} \cdot \hat{\beta}$$

The coordinate can be related by a **Frame Transform**.

$$\hat{\alpha} = K \cdot \hat{\beta}$$

$$\widehat{\alpha_\iota} = K_{ij} \widehat{\beta_\jmath}$$

Thus the transform of coordinate can be found by:

$$\vec{U} = u_i^\alpha \widehat{\alpha_\iota} = u_i^\alpha K_{ij} \widehat{\beta_\jmath} = u_j^\alpha K_{ji} \widehat{\beta_\iota} = u_i^\beta \widehat{\beta_\iota}$$

$$u_i^\beta = u_j^\alpha K_{ji} \leftrightarrow \overrightarrow{u_\beta} = \overrightarrow{u_\alpha} \cdot K$$

The matrix always acts on the right hand side to a coordinate. While the matrix acts on left hand side of a basis.

A summary in diagram

$$\begin{array}{cc|cc} & & & \vec{U} \\ K.\hat{\beta} = & \hat{\alpha} & \overrightarrow{u_\alpha} & \\ K & \uparrow & \downarrow & K \\ & \hat{\beta} & \overrightarrow{u_\beta} & = \overrightarrow{u_\alpha} \cdot K \end{array}$$

Vector Transform

A vector transform is based on a same frame.

$$\vec{U} = \overrightarrow{u_\alpha} \cdot \hat{\alpha} \xrightarrow{T} \vec{V} = T(\vec{U})$$

We use the bracket on the transform, since there is no matrix can act on a vector. The transform matrix defines the change of coordinate, *the transform matrix act on the coordinate, not the basis.* i.e.

$$\overrightarrow{v_\alpha} = \overrightarrow{u_\alpha} \cdot T$$

Thus, the vector transform can also be called **Coordinate Transform**. The vector transform is:

$$\vec{V} = \overrightarrow{v_\alpha} \cdot \hat{\alpha} = (\overrightarrow{u_\alpha} \cdot T) \cdot \hat{\alpha}$$

Some may like to think the transform as so change the basis:

$$\vec{V} = \overrightarrow{v_\alpha} \cdot \hat{\alpha} = \overrightarrow{u_\alpha} \cdot (T \cdot \hat{\alpha}) = \overrightarrow{u_\alpha} \cdot \hat{\gamma}$$

$$\hat{\gamma} = T \cdot \hat{\alpha}$$

Given that the new basis is also unitary and orthogonal.

And the coordinate of the transformed vector does not change but the basis.

$$\overrightarrow{u_\alpha} = \overrightarrow{v_\gamma}$$

And

$$\vec{V} = \overrightarrow{v_\alpha} \cdot \hat{\alpha} = \overrightarrow{v_\gamma} \cdot \hat{\gamma}$$

We can see, the transform T, is identical to the frame transform K. thus, a vector transform can also be regard as a frame transform.

A diagram of summery:

$$\begin{array}{ccc}
\vec{U} & \longrightarrow & \vec{V} = T\left(\vec{U}\right) \\
 & T & \\
\overrightarrow{u_\alpha} & \longrightarrow & \overrightarrow{v_\alpha} = \overrightarrow{u_\alpha} \cdot T
\end{array}$$

Vector transform and Frame transform

If we combine the diagram for both transform:

$$\begin{array}{ccc|cccccc}
 & & & \vec{U} & & \longrightarrow & & \vec{V} = T\left(\vec{U}\right) & \\
 & & & & & T & & & \\
K.\hat{\beta} = & \hat{\alpha} & & \overrightarrow{u_\alpha} & & \longrightarrow & \overrightarrow{v_\alpha} & = \overrightarrow{u_\alpha} \cdot T & \\
K & \uparrow & & \downarrow & K & & \downarrow & K & \\
 & \hat{\beta} & & \overrightarrow{u_\beta} & = \overrightarrow{u_\alpha} \cdot K & \longrightarrow & \overrightarrow{v_\beta} & = \overrightarrow{v_\alpha} \cdot K & \\
 & & & & & K^{-1} \cdot T \cdot K & & &
\end{array}$$

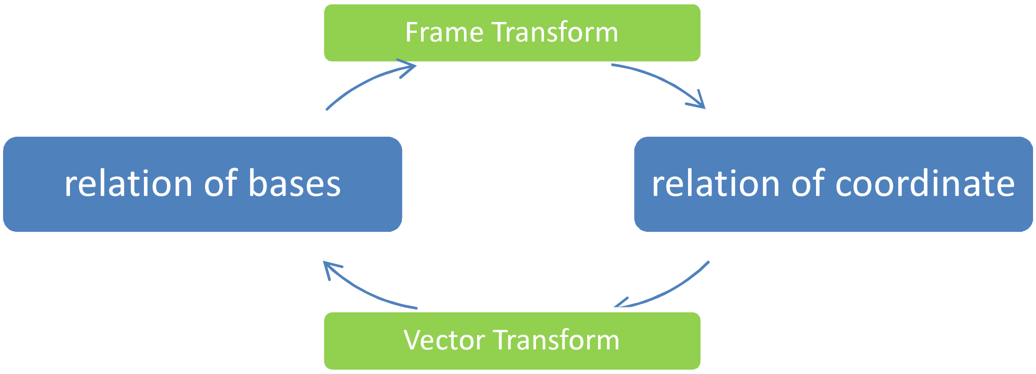
The coordinate transform in other frame is given by:

$$T_\beta = K^{-1} \cdot T \cdot K$$

Unless T and K **commutes**, they are different.

$$T_\beta = T \Leftrightarrow [T, K] = 0$$

We can see the different of vector transform and frame transform is the procedure. A frame transform is first defined the relation between bases, then find out the transform in coordinate. But the vector transform do the reverse.



If we like the Matrix act on the coordinate on right, then we replace

$$K \to G^T$$
$$T \to H^T$$

Then the row vector of coordinate will be in column vector.

$$\begin{array}{ccc|ccccc} & & & \vec{U} & & \underset{H}{\longrightarrow} & \vec{V} = H(\vec{U}) & \\ G^T.\hat{\beta} = & \hat{\alpha} & & \overrightarrow{u_\alpha} & & \longrightarrow & \overrightarrow{v_\alpha} & = H \cdot \overrightarrow{u_\alpha} \\ G & \uparrow & & \downarrow & G & & \downarrow & G \\ & \hat{\beta} & & \overrightarrow{u_\beta} & = G \cdot \overrightarrow{u_\alpha} & \underset{G \cdot H \cdot G^{-1}}{\longrightarrow} & \overrightarrow{v_\beta} & = G \cdot \overrightarrow{v_\alpha} \end{array}$$

The only ugly thing is the transform of the basis column vector.

Examples

In the example, we will use coordinate column vector. Or we us G & H

Rotation on z-axis

Both vector transform and frame transform on z-axis of the α-frame.

Frame transform

On the frame transform, the bases are related by:

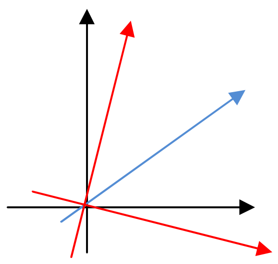
$$\begin{pmatrix} \widehat{\alpha_1} \\ \widehat{\alpha_2} \\ \widehat{\alpha_3} \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \widehat{\beta_1} \\ \widehat{\beta_2} \\ \widehat{\beta_3} \end{pmatrix} = G^T(\theta) \cdot \begin{pmatrix} \widehat{\beta_1} \\ \widehat{\beta_2} \\ \widehat{\beta_3} \end{pmatrix}$$

Or

$$G^{T}(\theta) \cdot \hat{\alpha} = \hat{\beta}$$

$$G(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is rotating the α-frame axis by positive θ to get the β-frame.



The black lines are axis of the α-frame, and red lines are axis of the β-frame. The blue line is the vector.

Any vector on the plane can be coordinated in this manner:

$$\vec{U} = \cos(\phi)\,\widehat{\alpha_1} + \sin(\phi)\,\widehat{\alpha_2} + 0\,\widehat{\alpha_3}$$

So, the coordinate is :

$$\overrightarrow{u_\alpha} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix}$$

According to the rule or direct calculation:

$$\overrightarrow{u_\beta} = \overrightarrow{u_\alpha} \cdot K(\theta) = \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ 0 \end{pmatrix}$$

The coordinate was rotated forward by θ degree in β-frame.

Remember, the vector does not change.

Vector transform

The coordinate transform is given by

$$H(\delta) = \begin{bmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overrightarrow{v_\alpha} = H(\delta) \cdot \overrightarrow{u_\alpha} = \begin{pmatrix} \cos(\phi + \delta) \\ \sin(\phi + \delta) \\ 0 \end{pmatrix}$$

The vector is

$$\vec{V} = \overrightarrow{v_\alpha} \cdot \hat{\alpha} = \cos(\phi + \delta)\,\widehat{\alpha_1} + \sin(\phi + \delta)\,\widehat{\alpha_2} + 0\,\widehat{\alpha_3}$$

And the vector does change. *Which is rotated the vector $\vec{U}$ by δ degree forward*. (well, since H is same as G) In most occasion, the discussion stop here.

Extra

If we keep the coordinate unchanged,

$$H(\delta) \cdot \hat{\alpha} = H^T(-\delta) \cdot \hat{\alpha} = \hat{\gamma}$$

This means, we have to rotate the α-frame forward by δ degree, in order to keep the coordinate unchanged. This can be understood by compensate the change of vector.

Combined

When combined, the coordinate of new vector $\vec{V}$ in basis β is:

$$\vec{V} = \overrightarrow{v_\beta} \cdot \hat{\beta} = G \cdot \overrightarrow{v_\alpha} \cdot \hat{\beta} = G \cdot H \cdot \overrightarrow{u_\alpha} \cdot \hat{\beta} = G \cdot H \cdot G^{-1} \cdot \overrightarrow{u_\beta} \cdot \hat{\beta}$$

This is the same construction from the diagram.

$$\overrightarrow{u_\beta} = \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ 0 \end{pmatrix}$$

$$G(\theta) \cdot H(\delta) \cdot G^{-1}(\theta) = \begin{bmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overrightarrow{v_\beta} = \begin{pmatrix} \cos(\phi + \theta + \delta) \\ \sin(\phi + \theta + \delta) \\ 0 \end{pmatrix}$$

Again, it is just rotated forward by δ degree.