Finite length Solenoid potential and field

The origin document was written in 2011. There are sign error and factor 2 error at that time. I update and redo the calculation in 2021. It is a supplementary document for the webpage https://nukephysik101.wordpress.com/2011/07/17/the-magnetic-field-of-a-finite-length-solenoid/. People should also refer to the calculation of the single coil. At the time of writing this document, I was a PhD student at Tokyo University.

Tsz Leung Tang, Saturday, July 10, 2021

The surface current density is (Jackson, 1998):

$$\vec{K} = \frac{I}{L}\delta(\rho - a)(-\sin\phi, \cos\phi, 0), \qquad z \in \left(-\frac{L}{2}, \frac{L}{2}\right).$$

In above, L is the total length of the solenoid, I is the current, (ρ, ϕ, z) are the cylindrical coordinate, and a is the radius of the solenoid. The general form of vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3x'.$$

By symmetry, the vector potential is azimuthal.

$$\vec{A} = A_\phi \hat{\phi} = \hat{\phi}\frac{\mu_0}{4\pi}\frac{I}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^{2\pi}\int_0^{\infty} \frac{\delta(\rho' - a)\cos\phi'}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2a\rho\cos\phi'}}\rho' d\rho' \, d\phi' dz',$$

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^{\pi} \frac{\cos\phi'}{\sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi'}} d\phi' dz'.$$

Simplify the form by setting $\zeta = (z - z')$ and the integration of ζ is a log function (Edmund E. Callaghan, 1960), we have,

$$\int_0^{\pi} \cos\phi' \left[\ln\left(\zeta + \sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}\right)\right]_{\zeta_-}^{\zeta_+} d\phi' = \left[\int_0^{\pi} \cos\phi' \ln\big(\zeta + \alpha(\zeta)\big)\, d\phi'\right]_{\zeta_-}^{\zeta_+},$$

$$\alpha(\zeta) = \sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}, \qquad \zeta_\pm = z \mp \frac{L}{2}.$$

Then,

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^{\pi} \cos\phi' \ln\big(\zeta + \alpha(\zeta)\big)\, d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Integration by path gives,

$$\int_0^{\pi} \cos\phi' \ln\big(\zeta + \alpha(\zeta)\big)\, d\phi' = \sin\phi' \ln\big(\zeta + \alpha(\zeta)\big)\big]_0^{2\pi} - \int_0^{\pi} \sin\phi'\, d\Big(\ln\big(\zeta + \alpha(\zeta)\big)\Big)$$

The first term is zero because of the $\sin\phi'$, and the derivative of $\ln\big(\zeta + \alpha(\zeta)\big)$ is:

$$\frac{d\ln\big(\zeta + \alpha(\zeta)\big)}{d\phi'} = \frac{\rho a \sin\phi'}{(\alpha(\zeta) + \zeta)\alpha(\zeta)}$$

Multiple by $(\alpha(\zeta) - \zeta)/(\alpha(\zeta) - \zeta)$

$$= \frac{(\alpha(\zeta) - \zeta)\rho a \sin\phi'}{(\alpha^2(\zeta) - \zeta^2)\alpha(\zeta)}$$
$$= \frac{\rho a \sin\phi'}{(\rho^2 + a^2 - 2\rho a \cos\phi')} - \frac{\zeta\rho a \sin\phi'}{(\rho^2 + a^2 - 2\rho a \cos\phi')\alpha(\zeta)}$$

The first term is a constant of ζ, and $[x]_{\zeta_-}^{\zeta_+} = 0$, then,

$$\int_0^{\pi} \sin\phi'\, d\Big(\ln\big(\zeta + \alpha(\zeta)\big)\Big) = -\int_0^{\pi} \frac{\zeta\rho a \sin^2\phi'}{(\rho^2 + a^2 - 2\rho a \cos\phi')\alpha(\zeta)} d\phi'$$

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia^2\rho}{L}\left[\zeta \int_0^{\pi} \frac{\sin^2\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a \cos\phi'}} d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Now, since the $\cos\phi$ is the same as interval $(-\pi, 0)$ and $(0, \pi)$, we can change the sign, and replacing $\phi = 2\theta$, using $\cos(2\theta) = 1 - 2\sin\theta$, we have

$$\int_0^{\pi} \frac{\sin^2\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a \cos\phi'}} d\phi'$$
$$= \int_0^{\pi} \frac{\sin^2\phi'}{(\rho^2 + a^2 + 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 + 2\rho a \cos\phi'}} d\phi'$$
$$= 2\int_0^{\frac{\pi}{2}} \frac{\sin^2(2\theta)}{((a+\rho)^2 - 4\rho a \sin^2\theta)\sqrt{\zeta^2 + (a+\rho)^2 - 4\rho a \sin^2\theta}} d\theta$$
$$= \frac{kh^2}{4\big(\sqrt{a\rho}\big)^3}\int_0^{\frac{\pi}{2}} \frac{\sin^2(2\theta)}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} d\theta$$
$$= \frac{kh^2}{\big(\sqrt{a\rho}\big)^3}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta - \sin^4\theta}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} d\theta, \qquad \sin^2(2\theta) = 4\sin^2\theta - 4\sin^4\theta$$
$$h^2 = \frac{4a\rho}{(a+\rho)^2}$$
$$k^2 = \frac{4a\rho}{(a+\rho)^2 + \zeta^2}$$

In the above calculation, in the 1st step, the $\cos\phi$ denominator change sign means the while function flipped horizontally on the $\phi = \frac{\pi}{2}$.

The integral can be spitted into 2 parts, the first part is (Milton Abramowitz, 1965) (NIST Digital Library of Mathematical Functions) :

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= -\frac{1}{h^2}\int_0^{\frac{\pi}{2}} \frac{1-h^2\sin^2\theta-1}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= -\frac{1}{h^2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}} - \frac{1}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{1}{h^2}\left(\Pi(h^2,k^2)-K(k^2)\right)
\end{aligned}$$

Here

$$\Pi(n,m) = \int_0^{\frac{\pi}{2}} \frac{1}{(1-n\sin^2\theta)\sqrt{1-m\sin^2\theta}}d\theta$$

It is the elliptic integral of 3rd kind.

The 2nd part is:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{-\sin^4\theta}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{1}{h^4}\int_0^{\frac{\pi}{2}} \frac{1-h^4\sin^4\theta-1}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{1}{h^4}\int_0^{\frac{\pi}{2}} \frac{1+h^2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{1}{(1-h^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{1}{h^4}\left(K(k^2)+\frac{h^2}{k^2}\left(K(k^2)-E(k^2)\right)-\Pi(h^2,k^2)\right)
\end{aligned}$$

Thus, combine everything and we have,

$$A_\phi = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\zeta k\left(\frac{k^2+h^2-h^2k^2}{h^2k^2}K(k^2)-\frac{1}{k^2}E(k^2)+\frac{h^2-1}{h^2}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

The magnetic field is the curl

$$\begin{aligned}
B_\rho &= \left[\nabla\times A_\phi\right]_\rho = -\frac{\partial}{\partial z}(A_\phi) = -\frac{\partial}{\partial\zeta}(A_\phi) \\
B_z &= \left[\nabla\times A_\phi\right]_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) = \frac{1}{\rho}A_\phi + \frac{\partial A_\phi}{\partial\rho}
\end{aligned}$$

Using the derivative formulae for elliptic integral:

$$\frac{d}{dk}K(k^2) = -\frac{1}{k}K(k^2) + \frac{1}{k(1-k^2)}E(k^2)$$
$$\frac{d}{dk}E(k^2) = -\frac{1}{k}K(k^2) + \frac{1}{k}E(k^2)$$
$$\frac{d}{dk}\Pi(h^2, k^2) = \frac{-k}{(1-k^2)(h^2-k^2)}E(k^2) - \frac{k}{(h^2-k^2)}\Pi(\nu^2, x^2)$$

Since derivative of ζ and ρ is through derivative of k, we compute,

$$\frac{d}{dk}\left(k\left(\frac{k^2+h^2-h^2k^2}{h^2k^2}K(k^2) - \frac{1}{k^2}E(k^2) + \frac{h^2-1}{h^2}\Pi(h^2,k^2)\right)\right)$$
$$= -\frac{1}{k^2}K(k^2) + \frac{h^2}{k^2(h^2-k^2)}E(k^2) + \frac{h^2-1}{(h^2-k^2)}\Pi(h^2,k^2)$$

And,

$$\frac{dk}{dz} = -\frac{k^3\left(z \pm \frac{L}{2}\right)}{4a\rho}, \qquad \frac{dk}{d\rho} = \frac{k}{2\rho} - \frac{k^3(a+\rho)}{4a\rho}$$

$$\frac{d}{dz}\big(zf(z)\big) = f(z) + z\frac{df(z)}{dz}, \qquad \frac{d}{d\rho}\left(\frac{1}{\sqrt{\rho}}f(\rho)\right) = -\frac{f(\rho)}{2\sqrt{\rho^3}} + \frac{1}{\sqrt{\rho}}\frac{df(\rho)}{d\rho}$$

Then

$$B_\rho = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{k}K(k^2) + \frac{2}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

Or, by using integration identity

$$\frac{d}{dx}\int_a^b f(x)dx = f(b) - f(a)$$

And the original formula of A_ϕ

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^\pi \frac{\cos\phi'}{\sqrt{\rho^2+a^2+(z-z')^2-2a\rho\cos\phi'}}d\phi' dz'.$$

We have,

$$B_\rho = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{\cos\phi'}{\sqrt{\zeta^2+\rho^2+a^2-2\rho a\cos\phi'}}d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Then, we can verify the two expression is the same. Our result for B_ρ is

$$B_\rho = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{k}K(k^2)+\frac{2}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}, \qquad k^2 = \frac{4a\rho}{(a+\rho)^2+\zeta^2}$$

To compute the z component, we need to be careful the $\Pi(h^2, k^2)$, since h^2 also contains ρ. And the z-component is:

$$B_z = \frac{\mu_0}{2\pi}\frac{I}{L}\frac{1}{2\sqrt{a\rho}}\left[\zeta k\left(K(k^2)+\frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

To verify the B_z, we can compute $\frac{\partial A_\phi}{\partial \rho}$ from

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \cos\phi' \ln\big(\zeta+\alpha(\zeta)\big)\,d\phi'\right]_{\zeta_-}^{\zeta_+}$$

By

$$\frac{\partial}{\partial\rho}\ln\big(\zeta+\alpha(\zeta)\big) = \frac{\rho - a\cos\phi'}{\alpha(\zeta)(\alpha(\zeta)+\zeta)}$$

Using the same trick

$$\frac{\rho - a\cos\phi'}{\alpha(\zeta)(\alpha(\zeta)+\zeta)} = \frac{(\rho - a\cos\phi')(\alpha(\zeta)-\zeta)}{\alpha(\zeta)(\alpha^2(\zeta)+\zeta^2)}$$
$$= \frac{(\rho - a\cos\phi')(\alpha(\zeta)-\zeta)}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')} = \frac{(\rho - a\cos\phi')}{(\rho^2+a^2-2\rho a\cos\phi')} - \frac{(\rho - a\cos\phi')\zeta}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')}$$

Therefore:

$$\frac{\partial A_\phi}{\partial\rho} = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{-\zeta\rho\cos\phi' + a\zeta\cos^2\phi'}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')}\,d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Combined with

$$\frac{1}{\rho}A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{a\zeta\sin^2\phi'}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')}\,d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Then the magnetic field is

$$B_z = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\zeta\int_0^\pi \frac{a-\rho\cos\phi'}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')}\,d\phi'\right]_{\zeta_-}^{\zeta_+}$$

With the same trick,

$$\int_0^\pi \frac{a-\rho\cos\phi'}{(\rho^2+a^2-2\rho a\cos\phi')\sqrt{\zeta^2+\rho^2+a^2-2\rho a\cos\phi'}}\,d\phi'$$

$$
\begin{aligned}
&= \int_0^{\pi} \frac{a + \rho\cos\phi'}{(\rho^2 + a^2 + 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 + 2\rho a\cos\phi'}} d\phi' \\
&= 2\int_0^{\frac{\pi}{2}} \frac{a + \rho\cos(2\theta)}{((a+\rho)^2 - 4\rho a\sin^2\theta)\sqrt{\zeta^2 + (a+\rho)^2 - 4\rho a\sin^2\theta}} d\theta \\
&= \frac{kh^2}{4\left(\sqrt{a\rho}\right)^3} \int_0^{\frac{\pi}{2}} \frac{(a+\rho) - 2\rho\sin^2\theta}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} d\theta \\
&= \frac{kh^2}{4\left(\sqrt{a\rho}\right)^3} \left((a+\rho)\,\Pi(h^2, k^2) - \frac{2\rho}{h^2}\left(\Pi(h^2,k^2) - K(k^2)\right) \right) \\
&= \frac{k}{4\left(\sqrt{a\rho}\right)^3} \left((h^2(a+\rho) - 2\rho)\Pi(h^2,k^2) + 2\rho K(k^2) \right) \\
&= \frac{k}{4\left(\sqrt{a\rho}\right)^3} \left(2\rho K(k^2) + \frac{2\rho(a-\rho)}{a+\rho}\Pi(h^2,k^2) \right) \\
&= \frac{k}{2a\sqrt{a\rho}} \left(K(k^2) + \frac{(a-\rho)}{a+\rho}\Pi(h^2,k^2) \right)
\end{aligned}
$$

Thus we get the same result.

$$
B_z = \frac{\mu_0}{2\pi}\frac{I}{L}\frac{1}{2\sqrt{a\rho}}\left[\zeta k\left(K(k^2) + \frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}
$$

In conclusion, the field is defined by:

$$
A_\phi = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\zeta k\left(\frac{k^2 + h^2 - h^2k^2}{h^2k^2}K(k^2) - \frac{1}{k^2}E(k^2) + \frac{h^2-1}{h^2}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}
$$

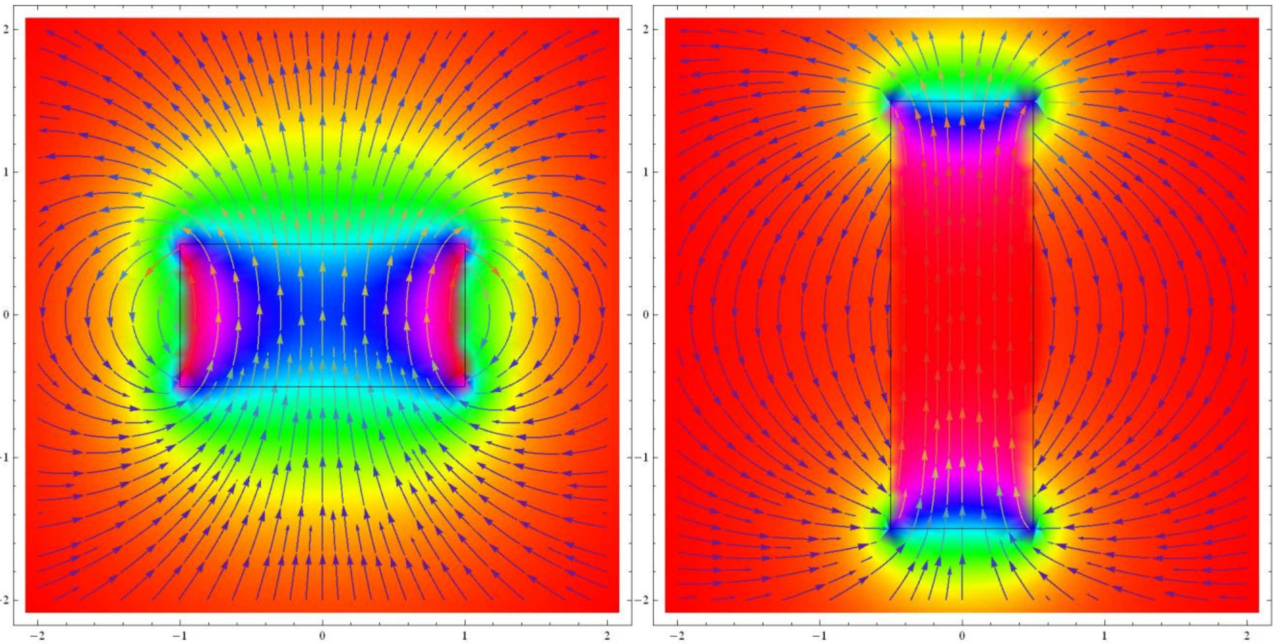
$$
B_\rho = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{k}K(k^2) + \frac{2}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}
$$

$$
B_z = \frac{\mu_0}{2\pi}\frac{I}{L}\frac{1}{2\sqrt{a\rho}}\left[\zeta k\left(K(k^2) + \frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}
$$

With

$$
h^2 = \frac{4a\rho}{(a+\rho)^2}, \qquad k^2 = \frac{4a\rho}{(a+\rho)^2 + \zeta^2}, \qquad \zeta_\pm = z \pm \frac{L}{2}
$$

Here are some plots:



Works Cited

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